Re-evaluation of Spin-Orbit Dynamics of Polarized e^+e^- Beams in High Energy Circular Accelerators and Storage Rings: Bloch equation approach [1]

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January 17, 2019
IAS Mini-Workshop on "Beam Polarization in Future Colliders"

[1] Work supported by the U.S. Department of Energy, Office of Science, Office of High Energy Physics, under Award Number DE-SC0018008

Outline-1

- Topic: Is polarization possible in high energy electron storage rings like proposed Circular Electron Positron Collider (CEPC) and Future Circular Collider (FCC-ee)?
- Review standard approach: Derbenev-Kondratenko formulas
- Derbenev-Kondratenko formulas rely, in part, on plausible assumptions grounded in deep physical intuition [2]
- Question: Do Derbenev-Kondratenko formulas, even with correction terms, tell full story?

[2] Ya.S. Derbenev, A.M. Kondratenko, *Sov. Phys. JETP*, vol. 37, p. 968, 1973.

Outline-2

- Alternative approach: Bloch equation for polarization density [3]
- Bloch equation allows for assessment of Derbenev-Kondratenko formulas
- Numerical approach to Bloch equation suggests Method of Averaging for getting effective Bloch equation
- Hope: Bloch equation teaches us domain of applicability of Derbenev-Kondratenko formulas

[3] Ya.S. Derbenev, A.M. Kondratenko, *Sov. Phys. Dokl.*, vol. 19, p. 438, 1975.

Derbenev-Kondratenko-formula approach-1

- Underlying model: [4] [5]
  1. Local polarization vector $\langle \vec{S} \rangle_{\theta,z}$ parallel to invariant spin field $\vec{n}(\theta, z)$:

$$\langle \vec{S} \rangle_{\theta,z} = \vec{n}(\theta, z) \left(\vec{n}(\theta, z) \cdot \langle \vec{S} \rangle_{\theta,z} \right) \tag{1}$$

  2. $\vec{n}(\theta, z) \cdot \langle \vec{S} \rangle_{\theta,z}$ independent of phase space position z:

$$\vec{n}(\theta, z) \cdot \langle \vec{S} \rangle_{\theta,z} \equiv P_{\mathrm{DK}}(\theta) \tag{2}$$

- Thus:

$$\langle \vec{S} \rangle_{\theta,z} = P_{\mathrm{DK}}(\theta) \vec{n}(\theta, z) \tag{3}$$

$$P_{\mathrm{DK}}(\theta) = P_{\mathrm{DK}}(+\infty)(1 - e^{-\theta/\tau_{\mathrm{DK}}}) + P_{\mathrm{DK}}(0) e^{-\theta/\tau_{\mathrm{DK}}} \tag{4}$$

where $\tau_{\mathrm{DK}}, P_{\mathrm{DK}}(+\infty)$ given by Derbenev-Kondratenko-formulas

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[4] S.R. Mane, Yu. M. Shatunov, and K. Yokoya, Rep. Prog. Phys. 68, 1997 (2005).

[5] D.P. Barber, G. Ripken, *Handbook of Accelerator Physics and Engineering*. Eds. A.W. Chao and M. Tigner, 1st edition, 3rd printing, World Scientific, 2006. See also arXiv:physics/9907034v2.

Derbenev-Kondratenko-formula approach-2

- Polarization vector $\vec{P}(\theta) \equiv$ phase space average of local polarization vector
- $\implies \vec{P}(\theta) = P_{\mathrm{DK}}(\theta)\langle\vec{n}\rangle_\theta$
- Polarization of electron bunch $\equiv \left|\vec{P}(\theta)\right| = \left|P_{\mathrm{DK}}(\theta)\right| \left|\langle\vec{n}\rangle_\theta\right|$
- Away from spin-orbit resonance: $\left|\langle\vec{n}\rangle_\theta\right| \approx 1$

Derbenev-Kondratenko-formula approach-3

- Invariant spin field $\vec{n}(\theta, z)$ satisfies T-BMT-equation in phase space:

$$\partial_\theta \vec{n} = \underbrace{L_{\text{Liou}}(\theta, z)\vec{n}}_{\text{Liouville terms}} + \underbrace{\vec{\Omega}(\theta, z) \times \vec{n}}_{\text{T-BMT-terms}} \tag{5}$$

with

1. $\left|\vec{n}(\theta, z)\right| = 1$
2. $\vec{n}(\theta + 2\pi, z) = \vec{n}(\theta, z)$

- Characteristic equation of (5) is T-BMT-equation:

$$\frac{d}{d\theta}\vec{n}(\theta, z(\theta)) = \vec{\Omega}(\theta, z(\theta)) \times \vec{n}(\theta, z(\theta))$$

Derbenev-Kondratenko-formula approach-4

- Derbenev-Kondratenko-formulas derived from semiclassical QED:

$$P_{\mathrm{DK}}(+\infty) = \frac{\tau_0^{-1}}{\tau_{\mathrm{DK}}^{-1}} \tag{6}$$

$$\tau_{\mathrm{DK}}^{-1} = \frac{5\sqrt{3}}{8}\frac{r_e\gamma^5\hbar}{m}\frac{C}{4\pi^2}\int_0^{2\pi} d\theta \frac{1}{|\rho(\theta)|^3}\left\langle 1 - \frac{2}{9}(\vec{n}\cdot\hat{\beta})^2 + \frac{11}{18}\left|\frac{\partial\vec{n}}{\partial z_6}\right|^2\right\rangle_\theta \tag{7}$$

1. $\tau_0^{-1} = \frac{r_e\gamma^5\hbar}{m}\frac{C}{4\pi^2}\int_0^{2\pi} d\theta \frac{1}{|\rho(\theta)|^3}\left\langle \hat{b}\cdot\left[\vec{n} - \frac{\partial\vec{n}}{\partial z_6}\right]\right\rangle_\theta$
2. z_6 =longitudinal momentum, $\hat{b}$ = normalized magnetic field, $\hat{\beta}$ = normalized velocity vector
3. Correction terms to τ_{DK}^{-1} corresponding to spin resonances are under debate [6]

[6] See, e.g., Z. Duan, M. Bai, D.P. Barber, Q. Qin, *A Monte-Carlo simulation of the equilibrium beam polarization in ultra-high energy electron (positron) storage rings*, Nucl. Instr. Meth. A793 (2015), pp.81-91. Available also at arXiv.

Derbenev-Kondratenko-formula approach-5

- Radiative depolarization rate:

$$\tau_{dep}^{-1} \equiv \frac{5\sqrt{3}}{8}\frac{r_e\gamma^5\hbar}{m}\frac{C}{4\pi^2}\int_0^{2\pi} d\theta \frac{1}{|\rho(\theta)|^3}\left\langle \frac{11}{18}\left|\frac{\partial\vec{n}}{\partial z_6}\right|^2\right\rangle_\theta \quad (8)$$

- $\Longrightarrow$ Neglecting spin-flip effects we get
  1. $P_{\mathrm{DK}}(+\infty) = 0$
  2. $P_{\mathrm{DK}}(\theta) = P_{\mathrm{DK}}(0)e^{-\theta/\tau_{\mathrm{DK}}}$

Bloch equation in lab frame-1

- Semiclassical quantum statistical approach
- Description of electron bunch by spin-$1/2$ Wigner function $\rho_{\rm lab}$:

$$\rho_{\rm lab}(t,r,p) = \frac{1}{2}[f_{\rm lab}(t,r,p)I_{2\times 2} + \vec{\sigma}\cdot\vec{\eta}_{\rm lab}(t,r,p)] \tag{9}$$

  1. $f_{\rm lab}$ = phase space density of bunch
  2. $\vec{\eta}_{\rm lab}$ = polarization density of bunch
- $\frac{\vec{\eta}_{\rm lab}}{f_{\rm lab}}$ =local polarization vector
- $\vec{P}_{\rm lab}(t) \equiv \int_{\mathbb{R}^6} drdp\vec{\eta}_{\rm lab}(t,r,p)$ = polarization vector of bunch
- $\left|\vec{P}_{\rm lab}(t)\right|$ = polarization of bunch
- $\vec{\sigma}$ = 3-vector of Pauli matrices

Bloch equation in lab frame-2

- Semiclassical QED$\Longrightarrow$ evolution law for $\rho_{\rm lab}$ [7] $\Longrightarrow$
  1. Fokker-Planck equation for phase space density:

$$\partial_t f_{\rm lab} = \underbrace{L_{\rm lab}(t,r,p) f_{\rm lab}}_{\text{Liouville \& damping \& diffusion}} \tag{10}$$

  2. Bloch equation for polarization density:

$$\partial_t \vec{\eta}_{\rm lab} = \underbrace{L_{\rm lab}(t,r,p)\vec{\eta}_{\rm lab}}_{\text{Liouville \& damping \& diffusion}} + \underbrace{\vec{\Omega}_{\rm lab}(t,r,p) \times \vec{\eta}_{\rm lab}}_{\text{T-BMT-terms}} + \underbrace{G_{\rm lab}(t,r,p)\vec{\eta}_{\rm lab} + \vec{g}_{\rm lab}(t,r,p) f_{\rm lab}}_{\text{spin-flip terms}} \tag{11}$$

- Bloch equation is PDE describing linear driven oscillator with damping and diffusion
- Task: Find equilibrium polarization vector $\vec{P}_{\rm lab}(\infty)$
- Bloch equation generalizes Baier-Katkov-Strakhovenko ODE to include phase-space effects [8]

[7] Ya.S. Derbenev, A.M. Kondratenko, *Sov. Phys. Dokl.*, vol. 19, p. 438, 1975.
[8] V.N. Baier, V.M. Katkov, V.M. Strakhovenko, *Sov. Phys. JETP*, vol. 31, p. 908, 1970.

RBE in lab frame

- Neglecting spin flip terms, Bloch equation simplifies to our reduced Bloch equation (=RBE):

$$\partial_t \vec{\eta}_{\rm lab} = \underbrace{L_{\rm lab}(t,r,p)\vec{\eta}_{\rm lab}}_{\text{Liouville \& damping \& diffusion}} + \underbrace{\vec{\Omega}_{\rm lab}(t,r,p) \times \vec{\eta}_{\rm lab}}_{\text{T-BMT-terms}} \quad (12)$$

- Radiative depolarization rate $\tau_{\rm dep}^{-1}$ can be studied via RBE
- RBE contains main numerical subtleties of Bloch equation
- RBE can be alternatively derived from Langevin equations [9] [10]

[9] K. Heinemann, D.P. Barber, Nucl. Instr. Meth. A463 (2001), p.62. Erratum-ibid.A469:294,2001.

[10] K. Heinemann, DESY-97-166. On archive at: arXiv:physics/9709025, 1997.

RBE in beam frame-1

- Transformation from lab frame to beam frame coordinates θ, z:
  1. Orbital Fokker-Planck equation:

$$\partial_\theta f = \underbrace{L(\theta, z; \epsilon) f}_{\text{Liouville \& damping \& diffusion}} \tag{13}$$

  2. RBE:

$$\partial_\theta \vec{\eta} = \underbrace{L(\theta, z; \epsilon)\vec{\eta}}_{\text{Liouville \& damping \& diffusion}} + \underbrace{\vec{\Omega}(\theta, z; \epsilon) \times \vec{\eta}}_{\text{T-BMT-terms}} \tag{14}$$

- Fact: Local polarization vector $\langle \vec{S} \rangle_{\theta,z}$ equal to $\frac{\vec{\eta}(\theta,z)}{f(\theta,z)}$
- Beam frame polarization vector $\vec{P}(\theta)$ of bunch:

$$\vec{P}(\theta) = \int_{\mathbb{R}^6} dz \vec{\eta}(\theta, z) \tag{15}$$

RBE in beam frame-2

- RBE can be alternatively derived from Langevin equations: [11]

$$Z' = (A(\theta) + \epsilon\delta A(\theta))Z + \underbrace{\sqrt{\epsilon}B(\theta)\xi(\theta)}_{\text{white noise term}} \tag{16}$$

$$\vec{S}' = \vec{\Omega}(\theta, Z; \epsilon) \times \vec{S} \equiv [\Omega_0(\theta) + \epsilon\omega(\theta, Z)]\vec{S} \tag{17}$$

- Spin-orbit probability density $\mathcal{P}$:

$$\int_V dz d\vec{s}\mathcal{P}(\theta, z, \vec{s}) \equiv \text{probability } \{(Z(\theta), \vec{S}(\theta)) \in V\} \tag{18}$$

- Polarization density $\vec{\eta}$ related to spin-orbit probability density

$$\vec{\eta}(\theta, z) = \int d\vec{s}\vec{s}\,\mathcal{P}(\theta, z, \vec{s}) \tag{19}$$

- $\mathcal{P}$ satisfies spin-orbit Fokker-Planck equation combining (16),(17)
- $\Longrightarrow$ Polarization density $\vec{\eta}$ satisfies RBE [12]

[11] J.A. Ellison, H. Mais, G. Ripken, in: "Handbook of Accelerator Physics and Engineering" , second edition, edited by A. W. Chao, K.H. Mess, Maury Tigner, F. Zimmermann 2013.

[12] K. Heinemann, D.P. Barber, Nucl. Instr. Meth. A463 (2001), p.62. Erratum-ibid.A469:294,2001.

Effective RBE-1

- RBE (14) numerically quite complex, e.g., L is θ-dependent
- Complex equations can often be approximated by effective equations
- Idea: Obtain effective RBE from effective Langevin equations using Method of Averaging
- How do we get effective Langevin equations?
- Answer: We apply rigorous Method of Averaging to Langevin equations [13]

[13] K. Heinemann, D. Appelö, D.P. Barber, O. Beznosov, J.A. Ellison, *Invited talk and paper, ICAP18, Key West*, Oct 19–23, 2018.

Effective RBE-2

- Step 1: Transform Langevin equations to standard form for averaging:
  1. Transform $Z, \vec{S}$ to $U, \vec{T}$ via $Z = \Psi(\theta)U$ and $\vec{S} = \Phi(\theta)\vec{T}$
  2. $\Psi' = A(\theta)\Psi$ and $\Phi' = \Omega_0(\theta)\Phi$
  3. $\implies$ transformed Langevin equations:

$$U' = \epsilon D(\theta)U + \underbrace{\sqrt{\epsilon}C(\theta)\xi(\theta)}_{\text{white noise term}} \tag{20}$$

$$\vec{T}' = \epsilon\mathfrak{D}(\theta, U)\vec{T} \tag{21}$$

- Step 2: Approximate $U, \vec{T}$ by $U_a, \vec{T}_a$ via Method of Averaging $\implies$ effective Langevin equations:

$$U_a' = \epsilon\bar{D}U_a + \underbrace{\sqrt{\epsilon}C_a\xi_a(\theta)}_{\text{white noise term}} \tag{22}$$

$$\vec{T}_a' = \epsilon\bar{\mathfrak{D}}(U_a)\vec{T}_a \tag{23}$$

$\implies U = U_a + \mathcal{O}(\epsilon), \vec{T} = \vec{T}_a + \mathcal{O}(\epsilon)$ for $0 \leq \theta \leq \mathcal{O}(1/\epsilon)$

Effective RBE-3

- Effective RBE:

$$\partial_\theta \vec{\eta}_a = \underbrace{[-\epsilon \sum_{j=1}^{6} \partial_{u_j} (\bar{D}u)_j}_{\text{Liouville \& damping}} + \underbrace{\frac{\epsilon}{2} \sum_{i,j=1}^{6} \bar{\mathcal{E}}_{ij} \partial_{u_i} \partial_{u_j}]\vec{\eta}_a}_{\text{diffusion}} + \underbrace{\bar{\mathfrak{D}}(u)\vec{\eta}_a}_{\text{T-BMT-terms}} \tag{24}$$

- Effective RBE (24) simple enough for numerical approach
- All coefficients of (24) θ-independent

Numerical algorithm for effective RBE-1

- Represent phase space vector U_a by 3 pairs of polar coordinates
- Fourier transform angle variables $\Longrightarrow$ Fourier coefficients are functions of time and radial variables
- Discretize radial variables on tensor grid of Chebychev points by Collocation Method [14] [15]
- Evolve Fourier modes via ODE system in θ
- Discretize ODE system (time stepping)
- Remarks:
  1. Collocation Method and Fourier expansion are spectral methods
  2. Time stepping is done in high order

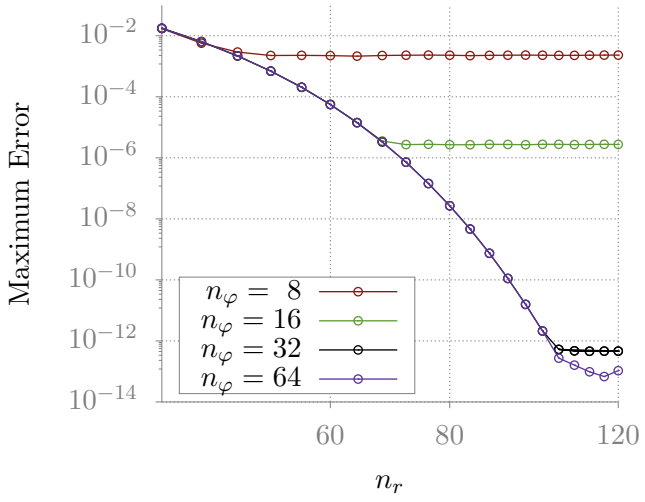
[14] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, *Spectral Methods. Fundamentals in Single Domains*, Springer, Berlin, 2006.

[15] B. Fornberg, *A Practical Guide to Pseudospectral Methods*, Cambridge University Press, Cambridge, 1996.

Numerical algorithm for effective RBE-2

For time evolution, high order IMEX ODE solver is used [16] [17]

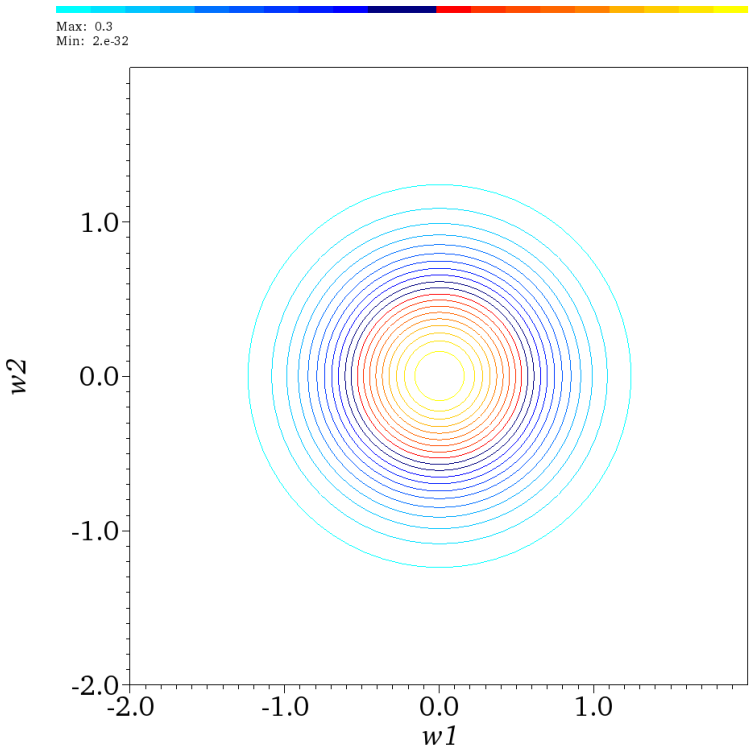
- High order methods like ours have high arithmetic intensity
- Accurate long time simulation is possible due to high parallel efficiency, robustness and spectral accuracy

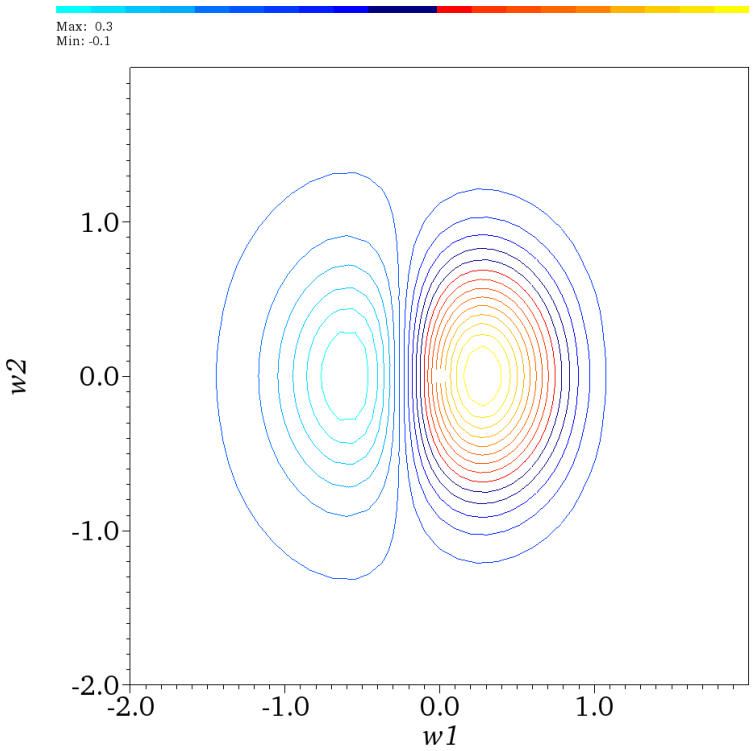


[16] O. Beznosov, D. Appelö, D.P. Barber, J.A. Ellison, K. Heinemann, *Talk and paper, ICAP18, Key West*, Oct 19–23, 2018.

[17] K. Heinemann, D. Appelö, D.P. Barber, O. Beznosov, J.A. Ellison, *Plenary talk and paper, ICAP18, Key West*, Oct 19–23, 2018.

Numerical algorithm for effective RBE-3





- Numerical solution approaches equilibrium
- For one degree of freedom, numerical solution matches the exact polarization obtained in [18] [19]
- Demonstration

[18] D.P. Barber, M. Böge, K. Heinemann, H. Mais, G. Ripken, Proc. 11th Int. Symp. High Energy Spin Physics, Bloomington, Indiana (1994)

[19] K. Heinemann, DESY-97-166. On archive at: arXiv:physics/9709025, 1997.

Future work

- Further development of Bloch equation approach (numerical and theoretical)
- Comparing the Bloch equation approach with Derbenev-Kondratenko-formula approach
- Better understanding/modification of Derbenev-Kondratenko-formula approach
  1. Study of correction term to τ_{DK}^{-1} in terms of RBE
  2. Replacing the invariant spin field $\vec{n}$ by "radiative invariant spin field" $\vec{p} \Longrightarrow \left|\frac{\partial \vec{n}}{\partial z_6}\right|^2$ replaced by $\left|\frac{\partial \vec{p}}{\partial z_6}\right|^2$ [20]

[20] D.P. Barber, K. Heinemann, DESY 15-142. On archive at: arXiv:1508.05318, 2015.